Standard and geometric approaches
to quantum Liouville theory on the pseudosphere [1]

Pietro Menotti

Dipartimento di Fisica dell'Università, Pisa 56100, Italy and
INFN, Sezione di Pisa
e-mail: menotti@df.unipi.it

Erik Tonni

Scuola Normale Superiore, Pisa 56100, Italy and
INFN, Sezione di Pisa
e-mail: e.tonni@sns.it

Abstract

We compare the standard and geometric approaches to quantum Liouville theory on the pseudosphere by performing perturbative calculations of the one and two point functions up to the third order in the coupling constant. The choice of the Hadamard regularization within the geometric approach leads to a discrepancy with the standard approach. On the other hand, we find complete agreement between the results of the standard approach and the bootstrap conjectures for the one point function and the auxiliary two point function.

Introduction

Classical Liouville theory is well understood even if it can be solved explicitly only in special cases. Quantum Liouville theory was first developed in the hamiltonian framework [1, 2]. Later, functional techniques were applied to the euclidean version of the theory.

Within the functional formulation there exist the so called standard approach [3, 4, 5, 6, 7, 8] and the geometric approach [9]. In the first case one introduces the vertex function by adding to the Liouville action external currents as is usually done in quantum field theory. Within the geometric approach one starts from the regularized classical action in presence of boundary terms that represent the sources and then considers the fluctuations of the field around the classical background.

In this paper we shall compare the standard and the geometric formulations of quantum Liouville theory on the sphere and the pseudosphere with particular attention to the pseudosphere [3], where the perturbative results are directly compared. Both formulations need to be regularized and such a regularization process is crucial, since the results depend non trivially on the adopted regularization procedure.

In the geometric approach the action is defined through a limit process and such a structure is not so easy to use in explicit calculations. However it is possible, by introducing a background field and a source field, to rewrite the action in a form such that no limit procedure appears. This structure is not as elegant as the original one, but from it one can read directly the transformation properties of the off shell action and consequently also of the correlation functions. This form can be used as a starting point for perturbative calculations of the correlation functions of vertex operators on the pseudosphere.

As first shown in [9], the appealing properties of the geometric action is to transform under conformal transformations as the vertex correlation functions of the quantum theory, generating quantum conformal dimensions $\Delta_\alpha = \alpha (Q - \alpha)$, which are those found in the hamiltonian approach, provided the central charges are properly identified.

On the sphere, due to the bounds imposed by the Picard inequalities, the situation is more complex and in this paper we shall perform perturbative computations only on the pseudosphere.

With regard to the one point function the main outcome will be the following: the results of the two formulations agree up to order b^4 included for the values of the first cumulant G_1, provided one properly identifies the coupling constant b_{g} of the geometric approach as a function of the coupling constant b of the standard approach and the same for the cosmological constants. In order to match the value of the second cumulant G_2,

one has to introduce by hand a coupling constant dependence in the source subtraction term of the geometric action. Such a b_g dependent subtraction influences only the second cumulant and has no effects on the other cumulants (G_1 and G_n with $n \geqslant 3$). A significant test of the equivalence of the two perturbative expansions can be achieved by computing the third cumulant G_3. This is easily done to first order, yet some improvement in the computational technique is needed both in the standard and geometric approach, in order to get the third order coefficient. The explicit computation of the third cumulant to order b^3 within the standard approach has been given in [10] and it disagrees with the result of the geometric approach computed in Section 2 of the present paper.

From the general field theoretical point of view we find that within the geometric approach the asymptotic value of the the vacuum expectation value of the Liouville field reproduces the classical background value. This does not happen within the standard approach, where the two asymptotic behaviors agree only qualitatively. On the other hand, within the standard approach the field $e^{2b\phi}$, which appears in the cosmological term, transforms like a (1, 1) primary field, while within the geometric approach the analogous field $e^{2b_\mathrm{g}\phi}$ has not such a transformation property because its quantum conformal dimensions are ($1-b_g^2, 1-b_g^2$). The different characters of the operators appearing in the cosmological terms of the two approaches has been already noticed by Takhtajan [11]. Here we find a difference respectively at the second and third order in the perturbative expansion of the second and third cumulant and these differences cannot be matched consistently by a redefinition of the coupling constants.

The obtained results can be compared to the perturbative expansion of the formula conjectured by Zamolodchikov and Zamolodchikov (ZZ) for the one point function on the pseudosphere [3]. Complete agreement has been found with the perturbative computation within the standard approach up to the third order [10]. To gain deeper insight, we shall move further through the perturbative computation of the two point function on the pseudosphere. This allows not only a better comparison between the two approaches but also to compare the results with another conjecture [3, 4, 7], i.e. the two point function when one vertex is the degenerate field $e^{-b\phi}$. The interest of this computation is that, by taking a proper ratio of two and one point functions, we can extract quantities not depending on the possible ambiguities in the subtraction terms of the geometric approach. Once more we find that the two point functions are different within the two approaches and that the perturbative results of the standard approach agree with the perturbative expansion of the formula conjectured through the bootstrap method.

A deeper inspection shows that the origin of the differences between the results obtained within the two approaches lies in the different regulators employed and not in the

2

way used to introduce the sources. Indeed, it can be shown that adopting the ZZ regulator both the approaches produce the same results, identifying exactly the couplings and the cosmological constants. The discrepancy in the structure of the unperturbed dimensions is matched by the different origin of the zero order in the second cumulant of the one point function. For all the other orders there is a one to one correspondence between the contributions.

1   Geometric action on the pseudosphere

The pseudosphere can be represented on the upper half plane or on the unit disk Δ. We shall use mostly the disk representation.

At the quantum level the geometry is encoded by the boundary condition at ∞, i.e. on the unit circle.

Within the geometric approach, we assume that the Liouville field ϕ behaves like

$$
\phi \ \simeq \ -\frac{Q}{2} \log{(\, 1 - z\bar{z}\,)^2} + O(1) \qquad\qquad |z| \to 1 \tag{1}
$$

$$
\phi \ \simeq \ -\alpha_n \log{|\, z - z_n\,|^2} + O(1) \qquad\qquad z \ \to\ z_n \ . \tag{2}
$$

The N point vertex functions are defined as follows [9, 11]

$$
\langle\, V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_N)\,\rangle \ = \ \Big\langle\, e^{2\alpha_1\phi(z_1)}\dots e^{2\alpha_N\phi(z_N)}\,\Big\rangle \ = \ \frac{\displaystyle\int_{\mathcal{C}(\Delta)} \mathcal{D}\,[\,\phi\,]\; e^{-S_{\Delta,\,N}[\phi]}}{\displaystyle\int_{\mathcal{C}(\Delta)} \mathcal{D}\,[\,\phi\,]\; e^{-S_{\Delta,\,0}[\phi]}} \tag{3}
$$

where $S_{\Delta,\,N}\,[\,\phi\,]$ is the geometric action on the pseudosphere with N sources

$$
\begin{aligned}
S_{\Delta,N}[\phi] \ = \ \lim_{\substack{\varepsilon\to 0 \\ r\to 1}} \Bigg\{ & \int_{\Delta_{r,\varepsilon}} \left[\frac{1}{\pi}\,\partial_z\phi\,\partial_{\bar{z}}\phi + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}}\phi}\right] d^2z \\
& - \frac{Q}{2\pi i} \oint_{\partial\Delta_r} \phi \left(\frac{\bar{z}}{1 - z\bar{z}}\, dz - \frac{z}{1 - z\bar{z}}\, d\bar{z}\right) + f(r, b_{\mathrm{g}}) \\
& - \frac{1}{2\pi i} \sum_{n=1}^{N} \alpha_n \oint_{\partial\gamma_n} \phi \left(\frac{dz}{z - z_n} - \frac{d\bar{z}}{\bar{z} - \bar{z}_n}\right) - \sum_{n=1}^{N} \alpha_n^2 \log \varepsilon_n^2 \Bigg\} \ . \tag{4}
\end{aligned}
$$

The points $z_n \in \Delta$, for $n = 1\dots N$, are the positions of the sources; the domains of integration are $\Delta_r = \{|z| \leqslant r\}$, $\gamma_n = \{|z - z_n| \leqslant \epsilon_n\}$ and $\Delta_{r,\varepsilon} = \Delta_r\backslash\bigcup_n \gamma_n$, while $f(r, b_{\mathrm{g}})$

3

is a numerical subtraction term. $S_{\Delta,0}[\phi]$ is the action (4) in absence of sources, which is formally equal to the action of the standard approach.

The equation of motion is the Liouville equation in presence of sources

$$\partial_z \partial_{\bar{z}} \phi = \pi b_{\mathrm{g}} \, \mu_{\mathrm{g}} e^{2 b_{\mathrm{g}} \phi} - \pi \sum_{n=1}^{N} \alpha_n \delta^2 (z - z_n) \,. \tag{5}$$

We decompose ϕ as the sum of two classical fields (ϕ_B and ϕ_0) and a quantum field ϕ_M

$$\phi = \phi_M + \phi_0 + \phi_B \tag{6}$$

with the background field ϕ_B having the asymptotics (1), i.e.

$$\phi_B \simeq -\frac{Q}{2} \log (1 - z\bar{z})^2 + c_{B,\Delta} + o \, (1 - |z|) \qquad |z| \to 1 \tag{7}$$

where $c_{B,\Delta}$ is a constant.

One could choose for the source field ϕ_0 a solution of the equation

$$\partial_z \partial_{\bar{z}} \phi_0 = -\pi \sum_{n=1}^{N} \alpha_n \, \delta^2 (z - z_n) \tag{8}$$

similarly to what has been done on the sphere (see [12] and Appendix).

Requiring that ϕ_0 vanishes on the unit circle, one gets

$$\phi_0 = -\sum_{n=1}^{N} \alpha_n \, \log \left| \frac{z - z_n}{1 - z\bar{z}_n} \right|^2 . \tag{9}$$

However, such ϕ_0 vanishes too slowly for $|z| \to 1$, giving rise to ill defined integrals in the perturbative calculation.

Thus, instead of ϕ_0 given by (9), we shall choose g_0 satisfying

$$\left(\partial_z \partial_{\bar{z}} - 2 \frac{1}{(1 - z\bar{z})^2} \right) g_0 = -\pi \sum_{n=1}^{N} \alpha_n \delta^2 (z - z_n) \,. \tag{10}$$

The solution of this equation is

$$g_0(z; z_1 \ldots z_N) = 2 \sum_{n=1}^{N} \alpha_n \, g(z, z_n) \tag{11}$$

being $g(z, z')$ the propagator [2, 3]

$$g(z, z') = -\frac{1}{2} \left(\frac{1 + \eta}{1 - \eta} \log \eta + 2 \right) \tag{12}$$

4

and $\eta(z, z')$ the $SU(1,1)$ invariant

$$\eta(z, z') = \left| \frac{z - z'}{1 - z\bar{z}'} \right|^2 \tag{13}$$

which is related to the geodesic distance between z and z'.

The source field g_0 converges to zero like $O\big((1 - z\bar{z})^2\big)$ when $|z| \to 1$, which makes the perturbative integrals convergent at infinity.

A procedure similar to the one employed for the sphere (see [12] and Appendix) gives the geometric action on the pseudosphere with a generic background field ϕ_B that satisfies the boundary conditions (7). The result is

$$
\begin{aligned}
S_{\Delta, N}[\phi] = {} & S_{\Delta,B}[\phi_B] + S_{\Delta,N,M}[\phi_M, \phi_B] + \sum_n^N \alpha_n \sum_{m \neq n}^N \alpha_m \left[\frac{1 + \eta_{n,m}}{1 - \eta_{n,m}} \log \eta_{n,m} + 2 \right] \\
& - \sum_n^N \alpha_n^2 \left[\log(1 - z_n\bar{z}_n)^2 - 2 \right] - 2 \sum_n^N \alpha_n \phi_B(z_n)
\end{aligned}
\tag{14}
$$

where $\eta_{n,m} = \eta(z_n, z_m)$ and $S_{\Delta,B}[\phi_B]$ is the background action

$$
\begin{aligned}
S_{\Delta,B}[\phi_B] = {} & \lim_{r \to 1} \left\{ \int_{\Delta_r} \left[\frac{1}{\pi} \partial_z \phi_B \, \partial_{\bar{z}} \phi_B + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}} \phi_B} \right] d^2 z \right. \\
& \left. - \frac{Q}{2\pi i} \oint_{\partial\Delta_r} \phi_B \left(\frac{\bar{z}}{1 - z\bar{z}} dz - \frac{z}{1 - z\bar{z}} d\bar{z} \right) + f(r, b_{\mathrm{g}}) \right\}
\end{aligned}
\tag{15}
$$

while $S_{\Delta,N,M}[\phi_M, \phi_B]$ is the action for the quantum field ϕ_M

$$
\begin{aligned}
S_{\Delta,N,M}[\phi_M, \phi_B] = {} & \int_\Delta \left[\frac{1}{\pi} \partial_z \phi_M \, \partial_{\bar{z}} \phi_M + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}} \phi_B} \left(e^{2b_{\mathrm{g}}(\phi_M + g_0)} - 1 \right) \right. \\
& \left. - \frac{2}{\pi}(\phi_M + g_0) \partial_z \partial_{\bar{z}} \phi_B - \mu_{\mathrm{g}} e^{2b_{\mathrm{g}} \phi_B^{cl}} 2 b_{\mathrm{g}}^2 g_0 (g_0 + 2\phi_M) \right] d^2 z \ .
\end{aligned}
\tag{16}
$$

The classical background field ϕ_B^{cl} is given by

$$\phi_B^{cl}(z) = -\frac{1}{2b_{\mathrm{g}}} \log \left[\pi b_{\mathrm{g}}^2 \mu_{\mathrm{g}} (1 - z\bar{z})^2 \right] \tag{17}$$

and it solves the Liouville equation

$$\partial_z \partial_{\bar{z}} \phi_B^{cl} = \pi b_{\mathrm{g}} \mu_{\mathrm{g}} e^{2b_{\mathrm{g}} \phi_B^{cl}} \tag{18}$$

with boundary conditions (7) if $Q = 1/b_{\mathrm{g}}$. The field ϕ_B^{cl} enters into $S_{\Delta,N,M}[\phi_M, \phi_B]$ because of the introduction of the singular field $g_0(z; z_1 \ldots z_N)$ through the equation (10). Under $SU(1,1)$ transformations

$$z \longrightarrow w = \frac{az + b}{\bar{b}z + \bar{a}} \qquad\qquad |a|^2 - |b|^2 = 1 \tag{19}$$

5

the background field transforms as follows

$$\phi_B(z) \quad \rightarrow \quad \phi'_B(w) = \phi_B(z) - \frac{Q}{2} \log \left| \frac{dw}{dz} \right|^2 \tag{20}$$

while ϕ_0 and ϕ_M are scalars. As a result, the background action (15) with $Q = 1/b_{\mathrm{g}}$ is $SU(1,1)$ invariant. The action $S_{\Delta,N,M}[\phi_M, \phi_B]$ is invariant as well and the terms that transform in (14) are

$$-\sum_n^N \alpha_n^2 \left[\log(1 - z_n \bar{z}_n)^2 \right] - 2 \sum_n^N \alpha_n \phi_B(z_n) . \tag{21}$$

Therefore the transformation law for the geometric action under $SU(1,1)$ is

$$S'_{\Delta,N}[\phi'] = S_{\Delta,N}[\phi] + \sum_{n=1}^N \alpha_n (Q - \alpha_n) \log \left| \frac{dw}{dz} \right|^2_{z=z_n} \tag{22}$$

where $Q = 1/b_{\mathrm{g}}$.

It is important to observe that such a trasformation property for the action holds off shell, i.e. also when ϕ is not a solution of the equation of motion.

With respect to the integration measure, we could choose the one induced by the distance

$$(\delta\phi, \delta\phi) = \int_\Delta |\delta\phi|^2 e^{2b_{\mathrm{g}}\phi} d^2z \tag{23}$$

or by the distance

$$(\delta\phi, \delta\phi) = \int_\Delta |\delta\phi|^2 e^{2b_{\mathrm{g}}\phi_B} d^2z . \tag{24}$$

Both are invariant under the group $SU(1,1)$, when ϕ_B transforms like (20) with $Q = 1/b_{\mathrm{g}}$, but the measure induced by (23) is not invariant under translation of the Liouville field ϕ, while the other one is.

The measure induced by (23) differs from the one induced by (24) by ultralocal terms. Such a difference should not be relevant in perturbative calculation [13]. We will work with (24), which gives rise to an integration measure that is invariant under translations in the field ϕ.

Thus, we have that

$$\int_{\mathcal{C}(\Delta)} \mathcal{D}[\phi] \, e^{-S_{\Delta,N,M}[\phi_M,\phi_B]} = \int_{\mathcal{C}(\Delta)} \mathcal{D}[\phi_M] \, e^{-S_{\Delta,N,M}[\phi_M,\phi_B]} \tag{25}$$

and it is invariant under $SU(1,1)$, for every N.

From the transformation law (22), one derives the quantum conformal dimensions of the Liouville vertex operators $e^{2\alpha\phi}$

$$\Delta_\alpha = \alpha (Q - \alpha) = \alpha \left(\frac{1}{b_{\mathrm{g}}} - \alpha \right) . \tag{26}$$

6

On the classical background $\phi_B = \phi_B^{cl}$ given in (17) $S_{\Delta,N,M}[\phi_M, \phi_B]$ can be written as follows

$$S_{\Delta,N,M}[\phi_M, \phi_B^{cl}] =$$

$$= \int_\Delta \left[\frac{1}{\pi} \partial_z \phi_M \partial_{\bar{z}} \phi_M + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}} \phi_B^{cl}} \left(e^{2b_{\mathrm{g}}(\phi_M + g_0)} - 1 - 2b_{\mathrm{g}}(\phi_M + g_0) \right) \right.$$

$$\left. - \mu_{\mathrm{g}} e^{2b_{\mathrm{g}} \phi_B^{cl}} 2b_{\mathrm{g}}^2 g_0 (g_0 + 2\phi_M) \right] d^2z$$

$$= \int_\Delta \left[\frac{1}{\pi} \partial_z \phi_M \partial_{\bar{z}} \phi_M + \frac{2}{\pi} \frac{\phi_M^2}{(1 - z\bar{z})^2} \right] d^2z + \sum_{k=3}^{\infty} \frac{1}{k!} \int_\Delta \frac{(2b_{\mathrm{g}}(\phi_M + g_0))^k}{\pi b_{\mathrm{g}}^2 (1 - z\bar{z})^2} d^2z$$

$$= S_{\Delta,0,M}[\phi_M, \phi_B^{cl}] + \widetilde{S}_{\Delta,N,M}[\phi_M, \phi_B^{cl}] \tag{27}$$

where
$$S_{\Delta,0,M}[\phi_M, \phi_B^{cl}] = \int_\Delta \left[\frac{1}{\pi} \partial_z \phi_M \partial_{\bar{z}} \phi_M + \frac{e^{2b_{\mathrm{g}} \phi_M} - 1 - 2b_{\mathrm{g}} \phi_M}{\pi b_{\mathrm{g}}^2 (1 - z\bar{z})^2} \right] d^2z \tag{28}$$

is formally identical to the action for the quantum field of the standard approach.
The source field $g_0(z; z_1, \ldots z_N)$ is contained only in $\widetilde{S}_{\Delta,N,M}[\phi_M, \phi_B^{cl}]$

$$\widetilde{S}_{\Delta,N,M}[\phi_M, \phi_B^{cl}] = \frac{2}{b_{\mathrm{g}}} \int_\Delta \frac{g_0}{\pi (1 - z\bar{z})^2} \left[e^{2b_{\mathrm{g}} \phi_M} - 1 - 2b_{\mathrm{g}} \phi_M \right] d^2z$$

$$+ 2 \int_\Delta \frac{g_0^2}{\pi (1 - z\bar{z})^2} \left[e^{2b_{\mathrm{g}} \phi_M} - 1 \right] d^2z$$

$$+ \frac{1}{b_{\mathrm{g}}^2} \sum_{k=3}^{\infty} \frac{(2b_{\mathrm{g}})^k}{k!} \int_\Delta \frac{g_0^k}{\pi (1 - z\bar{z})^2} \left[e^{2b_{\mathrm{g}} \phi_M} \right] d^2z . \tag{29}$$

From the second form of (27), we see that the propagator $g(z, z') = \langle \phi_M(z) \phi_M(z') \rangle$ in the geometric approach is the same as in the standard approach [3].
In order to compare the geometric approach with the standard one, the third form of (27) turns out to be the most useful.

2 One point function

In this section we shall provide the perturbative expansion of the one point function on the pseudosphere in the geometric approach and we shall compare it with the results of the standard approach [3, 10].

We recall that, within the geometric approach, the expectation value of the Liouville vertex operator $V_{\alpha_1}(z_1) = e^{2\alpha_1\phi(z_1)}$ is given by

$$\langle\, V_{\alpha_1}(z_1)\,\rangle = \Big\langle e^{2\alpha_1\phi(z_1)}\Big\rangle = \frac{\displaystyle\int_{\mathcal{C}(\Delta)}\mathcal{D}\,[\,\phi\,]\,e^{-S_{\Delta,1}[\phi]}}{\displaystyle\int_{\mathcal{C}(\Delta)}\mathcal{D}\,[\,\phi\,]\,e^{-S_{\Delta,0}[\phi]}} \tag{30}$$

being $S_{\Delta,1}\,[\,\phi\,]$ the geometric action (14) with one source of charge α_1.

To perform a perturbative calculation, we choose $\phi_B = \phi_B^{cl}$ given in (17), as done in [3]. Then the geometric action (14) with one source simplifies to

$$\begin{aligned}
S_{\Delta,1}[\phi] &= S_{\Delta,B}[\phi_B^{cl}] + S_{\Delta,1,M}[\phi_M, \phi_B^{cl}] \\
&\quad + \frac{\alpha_1}{b_{\mathrm{g}}}\,\log\big[\,\pi b_{\mathrm{g}}^2\mu\,(\,1-z_1\bar{z}_1\,)^2\,\big] - \alpha_1^2\,\big[\,\log\,(\,1-z_1\bar{z}_1\,)^2 - 2\,\big]
\end{aligned} \tag{31}$$

where $S_{\Delta,1,M}[\phi_M, \phi_B^{cl}]$ is the action (27) with $N = 1$ and the source field is given by $g_0(z; z_1) = 2\alpha_1 g(z, z_1)$.

The one point function (30) in the geometric approach can be written as

$$\langle\, V_{\alpha_1}(z_1)\,\rangle = \frac{U_{\mathrm{g}}}{(\,1-z_1\bar{z}_1\,)^{\,2\alpha_1(Q-\alpha_1)}} \tag{32}$$

where $Q = 1/b_{\mathrm{g}}$ and

$$\begin{aligned}
U_{\mathrm{g}} &= \big[\,\pi b_{\mathrm{g}}^2\mu_{\mathrm{g}}\,\big]^{-\alpha_1/b_{\mathrm{g}}}\,e^{-2\alpha_1^2}\,\frac{\displaystyle\int_{\mathcal{C}(\Delta)}\mathcal{D}\,[\,\phi_M\,]\,e^{-S_{\Delta,1,M}[\phi_M,\phi_B^{cl}]}}{\displaystyle\int_{\mathcal{C}(\Delta)}\mathcal{D}\,[\,\phi_M\,]\,e^{-S_{\Delta,0,M}[\phi_M,\phi_B^{cl}]}} \\
&= \big[\,\pi b_{\mathrm{g}}^2\mu_{\mathrm{g}}\,\big]^{-\alpha_1/b_{\mathrm{g}}}\,e^{-2\alpha_1^2}\,\Big\langle e^{-\widetilde{S}_{\Delta,1,M}}\Big\rangle_{z_1}.
\end{aligned} \tag{33}$$

We observe here that U_{g}, taken as a function of the charge α_1, has the correct normalization $U_{\mathrm{g}}(\alpha_1 = 0) = 1$ [3], being $\widetilde{S}_{\Delta,1,M} = 0$ when $\alpha_1 = 0$. Moreover, the mean value in the second line of (33) does not depend on μ_{g}.

As noticed in [11], one obtains the same central charge and the same quantum conformal dimensions of the standard approach by relating the coupling constant of the geometric approach b_{g} to the coupling constant of the standard approach b as follows

$$\frac{1}{b_{\mathrm{g}}} = \frac{1}{b} + b\,. \tag{34}$$

Thus, the comparison between the one point functions of the two theories reduces to the comparison between $U_{\mathrm{g}}(\alpha_1, b_{\mathrm{g}}, \mu_{\mathrm{g}})$ in the geometric approach and $U(\alpha_1, b, \mu)$ in the

standard approach [3].

Following [3], it is useful to consider, instead of $\langle e^{2\alpha_1 \phi(z_1)} \rangle$, the cumulant expansion

$$
\begin{aligned}
\log \langle e^{2\alpha_1 \phi(z_1)} \rangle &= \sum_{n=1}^{\infty} \frac{(2\alpha_1)^n}{n!} G_n^{\mathrm{g}} \\
&= 2\alpha_1 \left[-\frac{1}{2b_{\mathrm{g}}} \log (1 - z_1 \bar{z}_1)^2 \right] + \frac{(2\alpha_1)^2}{2} \left[\log (1 - z_1 \bar{z}_1) \right] + \log U_{\mathrm{g}} .
\end{aligned}
\tag{35}
$$

Using the expression of U_{g} given in (33), we get

$$
G_1^{\mathrm{g}} = \langle \phi(z_1) \rangle = -\frac{1}{2b_{\mathrm{g}}} \log (1 - z_1 \bar{z}_1)^2 - \frac{1}{2b_{\mathrm{g}}} \log [\pi b_{\mathrm{g}}^2 \mu_{\mathrm{g}}] - \frac{1}{2} \langle \widetilde{S}_{\Delta,1,M}^{(1)} \rangle
\tag{36}
$$

$$
\begin{aligned}
G_2^{\mathrm{g}} &= \langle \phi^2(z_1) \rangle - \langle \phi(z_1) \rangle^2 \\
&= \log (1 - z_1 \bar{z}_1) - 1 + \frac{1}{2^2} \left\{ -\langle \widetilde{S}_{\Delta,1,M}^{(2)} \rangle + \langle \left(\widetilde{S}_{\Delta,1,M}^{(1)} \right)^2 \rangle - \left(\langle \widetilde{S}_{\Delta,1,M}^{(1)} \rangle \right)^2 \right\}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
G_3^{\mathrm{g}} &= \langle \phi^3(z_1) \rangle - 3 \langle \phi^2(z_1) \rangle \langle \phi(z_1) \rangle + 2 \langle \phi(z_1) \rangle^3 \\
&= \frac{1}{2^3} \left\{ -\langle \widetilde{S}_{\Delta,1,M}^{(3)} \rangle + 3 \langle \widetilde{S}_{\Delta,1,M}^{(1)} \widetilde{S}_{\Delta,1,M}^{(2)} \rangle - 3 \langle \widetilde{S}_{\Delta,1,M}^{(1)} \rangle \langle \widetilde{S}_{\Delta,1,M}^{(2)} \rangle \right. \\
&\quad \left. - \langle \left(\widetilde{S}_{\Delta,1,M}^{(1)} \right)^3 \rangle + 3 \langle \widetilde{S}_{\Delta,1,M}^{(1)} \rangle \langle \left(\widetilde{S}_{\Delta,1,M}^{(1)} \right)^2 \rangle - 2 \left(\langle \widetilde{S}_{\Delta,1,M}^{(1)} \rangle \right)^3 \right\}
\end{aligned}
\tag{38}
$$

where

$$
\widetilde{S}_{\Delta,1,M}^{(k)} = \frac{\partial^k}{\partial \alpha_1^k} \widetilde{S}_{\Delta,1,M} \bigg|_{\alpha_1=0}
\tag{39}
$$

and the mean values are taken with respect to the action (28).

Using (36) and considering the $O(\alpha_1)$ contribution to (29), one obtains

$$
G_1^{\mathrm{g}} = \phi_B^{cl}(z_1) + \left[-\frac{1}{b_{\mathrm{g}}} \int_{\Delta} \frac{g_0^{(1)}(z; z_1)}{\pi(1 - z\bar{z})^2} \left(\langle e^{2b_{\mathrm{g}} \phi_M(z)} \rangle - 1 - 2b_{\mathrm{g}} \langle \phi_M(z) \rangle \right) d^2 z \right]
\tag{40}
$$

where $g_0^{(1)}(z; z_1) = 2 g(z, z_1)$ is the derivative with respect to α_1 of the classical source field $g_0(z; z_1)$.

The graphs contributing to G_1^{g} up to $O(b_{\mathrm{g}}^4)$ included are shown below, where the dashed lines represent $g_0^{(1)}(z; z_1)$.

9

$$G_1^{\mathrm{g}} \;=\; \phi_B^{cl}(z_1) \;+\; b_{\mathrm{g}} \left[\quad \bullet\text{--------}\bigcirc \quad \right]$$

$$+ \; b_{\mathrm{g}}^3 \left[\; \; \right]$$

$$+ \; O(b_{\mathrm{g}}^5) \; . \tag{41}$$

In the perturbative expansion divergent graphs appear due to the occurrence of $g(z,z)$; therefore the theory has to be regulated.

The proposal of [9] is to regulate such a propagator at coincident points with the Hadamard procedure (see also [14, 15, 16]), which amounts to set

$$g(z,z) \equiv \lim_{z' \to z} \left[\; g(z,z') + \frac{1}{2}\log\eta + \log 2 \; \right] \; . \tag{42}$$

This limit gives $g(z,z) = -1 + \log 2$; but for sake of generality we shall consider $g(z,z) = C$. We notice that within the geometric approach $g(z,z)$ has to be a constant in z, which implies U_{g} constant in z_1, otherwise the relation between the central charge

$$c_{\mathrm{g}} = 1 + 6\left(\frac{1}{b_{\mathrm{g}}}\right)^2 \tag{43}$$

and the quantum conformal dimensions of $\left\langle e^{2\alpha_1\phi(z_1)} \right\rangle$, which are

$$\Delta_{\alpha_1} = \alpha_1\left(\frac{1}{b_{\mathrm{g}}} - \alpha_1\right) \tag{44}$$

would be violated.

As already noticed in [11], expression (44) provides the cosmological term with quantum conformal dimensions $(\,1 - b_{\mathrm{g}}^2, 1 - b_{\mathrm{g}}^2\,)$.

The $O(b_{\mathrm{g}})$ contribution to G_1^{g} is given by $-b_{\mathrm{g}}\,C$. Concerning the $O(b_{\mathrm{g}}^3)$ contribution, we find that the three graphs contained in the second line of (41) sum up to zero and the same happens for the two graphs of the third line. The whole $O(b_{\mathrm{g}}^3)$ contribution comes from the two graphs of the fourth line: their values are $(\pi^2 - 15)/18$ and $2/3$ respectively. Thus, the expansion of G_1^{g} in the coupling constant b_{g} within the geometric approach, up to $O(b_{\mathrm{g}}^4)$ included, is

$$G_1^{\mathrm{g}} = -\frac{1}{2b_{\mathrm{g}}}\,\log\left[\,\pi b_{\mathrm{g}}^2 \mu_{\mathrm{g}}\,(\,1 - z_1 \bar{z}_1\,)^2\,\right] - b_{\mathrm{g}}\,C + b_{\mathrm{g}}^3\left(\frac{\pi^2}{18} - \frac{1}{6}\right) + O(b_{\mathrm{g}}^5)\,. \qquad (45)$$

Notice that, because of the cancellation between graphs explained just above, the regulator C appears only in the contribution $O(b_{\mathrm{g}})$. The contribution of the term in the square brackets of (40) is given by $\langle\,\phi_M(z_1)\,\rangle$. Indeed, varying ϕ_M in (28), we obtain for the expectation value of the equation of motion

$$-\frac{1}{\pi}\,\partial_z \partial_{\bar{z}}\,\langle\,\phi_M\,\rangle + \mu_{\mathrm{g}} b_{\mathrm{g}} e^{2b_{\mathrm{g}}\phi_B^{cl}}\left(\langle\,e^{2b_{\mathrm{g}}\phi_M}\,\rangle - 1\right) = 0 \qquad (46)$$

which can be rewritten, using the equation of the propagator and the expression (17), as follows

$$\langle\,\phi_M(z_1)\,\rangle = -\frac{2}{b_{\mathrm{g}}}\int_\Delta \frac{g(z_1, z)}{\pi(\,1 - z\bar{z}\,)^2}\left(\langle\,e^{2b_{\mathrm{g}}\phi_M(z)}\,\rangle - 1 - 2b_{\mathrm{g}}\langle\,\phi_M(z)\,\rangle\right)d^2z\,. \qquad (47)$$

Being $g_0^{(1)}(z; z_1) = 2\,g(z, z_1)$, we recognize in the term in square brackets of (40) the r.h.s. of the previous Ward identity.

Equation (47) has the same form of the Ward identity obtained by ZZ within the standard approach [3] for the quantum field χ and verified up to $O(b^3)$ with their regulator.

Direct computation of $\langle\,\phi_M(z_1)\,\rangle$ with the C (Hadamard) regulator up to b_{g}^4 included agrees with the r.h.s. of (45). On the other hand, we have checked the validity of the equation of motion (46) in the C regularized theory up to b_{g}^4 included. Indeed, (47) follows directly from the independence of $\langle\,\phi_M(z)\,\rangle$ on z and from $\langle\,e^{2b_{\mathrm{g}}\phi_M}\,\rangle = 1$. The perturbative expansion (45) gives the constancy of $\langle\,\phi_M(z)\,\rangle$ up to b_{g}^4 included, while we have checked that, with the C regulator, $\langle\,e^{2b_{\mathrm{g}}\phi_M}\,\rangle = 1$ is valid up to b_{g}^5 included. We expect such an identity to be valid to all orders in the coupling constant b_{g}.

Therefore we obtain up to b_{g}^4 included

$$G_1^{\mathrm{g}} = \langle\,\phi(z_1)\,\rangle = \phi_B^{cl}(z_1) + \langle\,\phi_M(z_1)\,\rangle\,. \qquad (48)$$

Notice that, due to (47), within the geometric approach $\langle\,\phi_M(z_1)\,\rangle$ cannot depend on the position z_1 if we want to keep for Q the value $1/b_{\mathrm{g}}$. As seen above this requirement is

satisfied by using the $SU(1,1)$ invariant regulator C [9].

The structure (48) was obtained also in the standard approach by [3], with the difference that $\langle \phi_M(z_1) \rangle$ is not z_1 independent because of the different choice of the regulator. In the standard approach that choice of regulator is necessary in order to provide the correct quantum conformal dimensions of the hamiltonian treatment [1]

$$\Delta_{\alpha_1} = \alpha_1 \left(\frac{1}{b} + b - \alpha_1 \right) . \tag{49}$$

Dependence on z_1 of G_1^{g} given in (45) compared to that of the standard approach [3] imposes the relation (34) between b_{g} and b, while the agreement between the constant terms can be obtained by choosing μ_{g} as a proper function of μ, b and C.

Concerning G_2^{g}, from (37) we get

$$
\begin{aligned}
G_2^{\mathrm{g}} = {} & \log\left(1 - z_1\bar{z}_1\right) - 1 - \int_\Delta \frac{(g_0^{(1)}(z;z_1))^2}{\pi(1-z\bar{z})^2} \left(\langle e^{2b_{\mathrm{g}}\phi_M(z)} \rangle - 1 \right) d^2z \\
& + \frac{1}{b_{\mathrm{g}}^2} \int_\Delta \frac{g_0^{(1)}(z;z_1)}{\pi(1-z\bar{z})^2} \frac{g_0^{(1)}(z';z_1)}{\pi(1-z'\bar{z}')^2} \left\langle \left(e^{2b_{\mathrm{g}}\phi_M(z)} - 1 - 2b_{\mathrm{g}}\phi_M(z) \right) \right. \\
& \qquad\qquad\qquad\qquad \left. \left(e^{2b_{\mathrm{g}}\phi_M(z')} - 1 - 2b_{\mathrm{g}}\phi_M(z') \right) \right\rangle d^2z\, d^2z' \\
& - \left(-\frac{1}{b_{\mathrm{g}}} \int_\Delta \frac{g_0^{(1)}(z;z_1)}{\pi(1-z\bar{z})^2} \left(\langle e^{2b_{\mathrm{g}}\phi_M(z)} \rangle - 1 - 2b_{\mathrm{g}}\langle \phi_M(z) \rangle \right) d^2z \right)^2 .
\end{aligned}
\tag{50}
$$

Through the Ward identity (47), the last term reduces to $\langle \phi_M(z_1) \rangle^2$, which has already been computed. The graphs that contribute to G_2^{g} are given below

$$G_2^{\mathrm{g}} = \log\left(1 - z_1\bar{z}_1\right) - 1$$

$$+ \, O(b_{\mathrm{g}}^4) . \tag{51}$$

12

The first integral in (50) gives the two graphs in the first line of the $O(b_g^2)$ contribution and they cancel out. Indeed, it can be shown that this integral vanishes at least up to the order b_g^5 included, in agreement with the already discussed Ward identity (47).

The second integral in (50) provides a nonvanishing $O(b_g^2)$ contribution through the first two graphs in the second line, but the second of these graphs simplifies with the third one, which is the order $O(b_g^2)$ of the last integral in (50), i.e. $\langle \phi_M(z_1) \rangle^2$.

Thus

$$G_2^{\mathrm{g}} = \log\left(1 - z_1\bar{z}_1\right) - 1 + b_g^2 \left(\frac{5}{6} - \frac{\pi^2}{18}\right) + O(b_g^4) \tag{52}$$

and it does not depend on C.

Since b_g is fixed by (34), G_2^{g} disagrees with the result of the standard approach [3]

$$G_2 = \log\left(1 - z_1\bar{z}_1\right) - 1 + b^2 \left(\frac{3}{2} - \frac{\pi^2}{6}\right) + O(b^4) \ . \tag{53}$$

Notice that the $O(b_g^0)$ contribution to G_2^{g} comes from the regularization of the geometric action, while in the standard approach the $O(b^0)$ contribution to G_2 is the result of the ZZ regularization of the one loop graph that contributes to this order.

One could modify the geometric action (4) by replacing the subtraction term $\sum_n \alpha_n^2 \log \varepsilon_n^2$ with $\sum_n \alpha_n^2 \log\left(\lambda_n^2 \varepsilon_n^2\right)$ with a properly chosen b_g dependent λ_n, i.e. $\lambda_n(b_g)$. This change would remove the discrepancy between G_2^{g} and G_2; but still differences remain both in the comparison between G_3^{g} and G_3, as we shall show in what follows, and in the comparison of the two point functions, as we shall discuss in the next section.

The graphs contributing to G_3^{g} in the geometric approach are shown below

$$
G_3^{\mathrm{g}} = b_{\mathrm{g}} \left[\; \text{} \; \right]
$$

$$
+ \; O(b_{\mathrm{g}}^5) \; .
$$

$$(54)$$
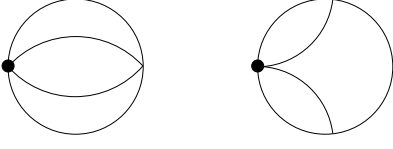
The $O(b_{\mathrm{g}})$ contribution to G_3^{g} is $-b_{\mathrm{g}}$. The graphs contributing to the order b_{g}^3 are shown in the big square brackets of (54). Adopting the Hadamard regularization, the two graphs in the second line sum up to zero and the same happens for the two ones in the third line. Thus, using partially the computation given in [10], we get the perturbative expansion of G_3^{g}, which is

$$
G_3^{\mathrm{g}} = - b_{\mathrm{g}} + b_{\mathrm{g}}^3 \left(\frac{3}{2} + \frac{\pi^2}{6} - 2\,\zeta(3) \right) + O(b_{\mathrm{g}}^5)
$$

$$(55)$$

On the other hand, in the standard approach one gets [10]

$$
G_3 = - b + b^3 \left(3 - 2\,\zeta(3) \right) + O(b^5)
$$

$$(56)$$

14

which is in contrast with (55), taking into the account the relation (34) between b and b_{g}.
As a further check, we have considered the first perturbative order of the fourth cumulant.
Within the standard approach the $O(b^2)$ contribution to G_4 is given by

$$
G_4 \;=\; b^2 \left[\begin{array}{cc} & \end{array} \right] + O(b^4) \;=\; -2\,b^2 + O(b^4) \quad (57)
$$

and it agrees with the perturbative expansion of G_4^{g} up to this order.

Instead of regulating the theory by giving a finite value to $g(z, z)$, one can proceed through the usual Pauli-Villars regulator technique, i.e. by introducing a regulator field ζ and replacing ϕ with $\phi + \zeta$ in the interaction lagrangian. We show here that the C and ZZ regulators arise from two Pauli-Villars regulators whose lagrangians possess different transformation properties.

Let us consider the following quadratic lagrangian

$$
-\frac{1}{\pi}\, \partial_z \zeta\, \partial_{\bar{z}} \zeta \;-\; \frac{1}{\pi}\, \frac{m\,(m-1)}{(1-z\bar{z})^2}\, \zeta^2 \tag{58}
$$

and the equation for the Green function that follows from it

$$
\frac{2}{\pi}\, \partial_z \partial_{\bar{z}}\, g_\zeta(z, z') \;-\; \frac{2}{\pi}\, \frac{m\,(m-1)}{(1-z\bar{z})^2}\, g_\zeta(z, z') \;=\; \delta^2(\,z - z'\,) \;. \tag{59}
$$

Such an equation is invariant under $SU(1,1)$. The explicit form of the Green function $g_\zeta(z, z')$ is

$$
g_\zeta(z, z') = -\frac{1}{2}\, \frac{\Gamma(m)^2}{\Gamma(2m)}\, (\,1 - \eta\,)^m \; {}_2F_1(m, m, 2m; 1 - \eta\,) \tag{60}
$$

where $\eta(z, z')$ is the invariant (13) and ${}_2F_1$ is the hypergeometric function.

Note that, for fixed $\eta \neq 0$, $g_\zeta(z, z') \to 0$ when $m \to \infty$. Because of this property, all the diagrams containing $g_\zeta(z, z')$ with $z \neq z'$ vanish when $m \to \infty$.

The divergence of $g_\zeta(z, z')$ at coincident points ($z \to z'$, i.e. $\eta \to 0$) cancels out the one of $g(z, z')$ as follows

$$
\lim_{z' \to z} \big[\, g(z, z') + g_\zeta(z, z')\,\big] \;=\; -1 + \gamma + \psi(m) \tag{61}
$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

Thus the Pauli-Villars regulator (60) generates a C regulator, where the constant C

diverges as $\log(m)$ when $m \to +\infty$.

A different Pauli-Villars regulating field can be introduced through the lagrangian

$$-\frac{1}{\pi}\partial_z\zeta\,\partial_{\bar{z}}\zeta \;-\; \frac{M^2}{4\pi}\,\zeta^2 \tag{62}$$

that gives the following equation for the Green function

$$\frac{2}{\pi}\,\partial_z\partial_{\bar{z}}\,g_\zeta(z,z') \;-\; \frac{M^2}{2\pi}\,g_\zeta(z,z') \;=\; \delta^2(\,z-z'\,)\;. \tag{63}$$

If we set

$$g_\zeta(z,z') = -K_0(\,2M\,|\,z-z'\,|\,) + R(z,z') \tag{64}$$

being K_0 the modified Bessel function of the second kind, then $R(z,z')$ satisfies the homogeneous elliptic equation

$$4\,\partial_z\partial_{\bar{z}}\,R(z,z') \;-\; M^2\,R(z,z') \;=\; 0 \tag{65}$$

with boundary condition

$$R(z,z') = K_0(\,2M\,|\,z-z'\,|\,) \qquad |z| \to 1 \tag{66}$$

that gives $g_\zeta(z,z') \to 0$ when $|z| \to 1$.

Using the maximum principle for this kind of equations [17], we obtain

$$|\,R(z,z')\,| \;\leqslant\; \max_{|w|=1}\; K_0(\,2M\,|\,w-z'\,|\,) \tag{67}$$

for every fixed z and z' inside the disk Δ.

Thus, for fixed z and z' inside the unit disk, we have that $R(z,z') \to 0$ when $M \to +\infty$, being $K_0(\,M\,|w-z'\,|\,) \to 0$ for $M \to +\infty$.

The field ζ provides the regulating field and, in this case, we find

$$\lim_{z'\to z}\,[\,g(z,z') + g_\zeta(z,z')\,] \;=\; \log\,(\,1-z\bar{z}\,) - 1 + \gamma + \log M + R(z,z)\;. \tag{68}$$

This reproduces the ZZ regularization [3] by a proper subtraction of the divergent term since, for every fixed z inside the disk, $R(z,z) = O(e^{-M})$.

In closing this section, we notice that, using (14), the decomposition of $S_{\Delta,N,M}[\phi_M,\phi_B^{cl}]$ given in (27) and the expression for the one point function (32), one can write the N point vertex functions (3) as follows

$$\langle\,V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\,\rangle = \left[\,\prod_{n=1}^{N}\langle\,V_{\alpha_n}(z_n)\,\rangle\,\right]\,e^{2\sum_n \alpha_n \sum_{m\neq n}\alpha_m\,g(z_n,z_m)}\,\frac{\Big\langle e^{-\widetilde{S}_{\Delta,N,M}}\Big\rangle_{z_1,\ldots,z_N}}{\prod_{n=1}^{N}\Big\langle e^{-\widetilde{S}_{\Delta,1,M}}\Big\rangle_{z_n}} \tag{69}$$

16

where the mean values in the ratio are taken with respect to (28).

If we compute formally the limit of the previous expression e.g. for $|z_1| \to 1$, we can verify the cluster decay at large distance

$$\lim_{|z_1| \to 1} \langle\, V_{\alpha_1}(z_1) \ldots V_{\alpha_N}(z_N)\,\rangle = \langle\, V_{\alpha_1}(z_1)\,\rangle\, \langle\, V_{\alpha_2}(z_2) \ldots V_{\alpha_N}(z_N)\,\rangle \tag{70}$$

that is the boundary condition used by ZZ [3] to get the one point function through the bootstrap method.

3   Two point function

To gain further insight into the relation between the two approaches, we provide the perturbative computation of the two point function.

First we compute the complete one loop order and some results to two loop within the standard approach for generic α_1 and α_2. Then, we compare these results with the ones obtained within the geometric approach for generic α_1 and α_2.

A perturbative check of the exact formula conjectured for the auxiliary two point function $\langle\, V_\alpha(z')\, V_{-b/2}(z)\,\rangle$ [3, 4, 7] will be given at the end of this section.

As pointed out in [3], it is more efficient to compute the ratio

$$g_{\alpha_1,\alpha_2}(\eta) = \frac{\langle\, V_{\alpha_1}(z_1)\, V_{\alpha_2}(z_2)\,\rangle}{\langle\, V_{\alpha_1}(z_1)\,\rangle\, \langle\, V_{\alpha_2}(z_2)\,\rangle} \tag{71}$$

where $\langle\, V_{\alpha_1}(z_1)\, V_{\alpha_2}(z_2)\,\rangle$ represents the full correlator and not only the connected component.

Taking into account once more the cumulant expansion

$$\log\left[\, g_{\alpha_1,\alpha_2}(\eta)\,\right] = \sum_{k_1,k_2=1}^{\infty} \frac{(\,2\alpha_1)^{k_1}}{k_1!}\frac{(\,2\alpha_2)^{k_2}}{k_2!}\, M_{k_1,k_2}(\eta) \tag{72}$$

we compute perturbatively $M_{k_1,k_2}(\eta)$.

From (71), it can be easily seen that the background field does not contribute to $g_{\alpha_1,\alpha_2}(\eta)$.

The graphs contributing to $M_{1,1}$ up to one loop are

$$M_{1,1} \quad = \quad \left[\; \text{} \; \right]$$

$$+ \quad O(b^4) \, .$$

(73)

As requested by the standard approach [3], we have computed them by using the ZZ regulator, obtaining

$$M_{1,1} \quad = \quad \langle \chi(z_1)\, \chi(z_2) \rangle - \langle \chi(z_1) \rangle \, \langle \chi(z_2) \rangle$$

$$= \quad g(\eta) + b^2 \left(\frac{3}{2} + \frac{\eta^2 \, \log^2 \eta}{2 \, (1-\eta)^2} - \frac{1+\eta}{1-\eta} \, \mathrm{Li}_2(1-\eta) \right) + O(b^4) \, .$$

(74)

As expected, the limit of the $O(b^2)$ contribution to $M_{1,1}$ for $\eta \to 0$ gives exactly the order $O(b^2)$ of G_2.

Concerning $M_{2,1}$, its first order expansion is given by the following graph

$$M_{2,1} \quad = \quad b \left[\; \text{} \; \right] \quad + \quad O(b^3)$$

(75)

and it gives [3]

$$M_{2,1} \quad = \quad \langle \chi^2(z_1)\, \chi(z_2) \rangle - \langle \chi^2(z_1) \rangle \, \langle \chi(z_2) \rangle$$

$$- 2 \, \langle \chi(z_1)\, \chi(z_2) \rangle \, \langle \chi(z_1) \rangle + 2 \, \langle \chi(z_1) \rangle^2 \, \langle \chi(z_2) \rangle$$

$$= \quad b \left(\frac{\eta \, \log^2 \eta}{(1-\eta)^2} - 1 \right) + O(b^3) \, .$$

(76)

We have considered also the two loop graphs that give the first order of $M_{3,1}$

$$M_{3,1} \quad = \quad b^2 \left[\; \text{} \; \right] \quad + \quad O(b^4) \, .$$

(77)

18

The result is

$$M_{3,1} = b^2 \left(-\frac{\eta\,(1+\eta)\,\log^3\eta}{(1-\eta)^3} - 2 \right) + O(b^4)\ . \tag{78}$$

To compute the graphs contributing to $O(b^2)$ in $M_{1,1}$ and to $O(b^2)$ in $M_{3,1}$, we have employed the same technique developed in [10], first giving (in analogy to the Gegenbauer method; see e.g. [18]) an harmonic expansion of the Green function (12) and then reducing the graphs, through angular integrations, to radial integrals.

Within the geometric approach, by using (69) with $N=2$, we introduce $g^{\mathrm{g}}_{\alpha_1,\alpha_2}(\eta)$ as we have done in (71) for the standard approach, obtaining

$$g^{\mathrm{g}}_{\alpha_1,\alpha_2}(\eta) = \frac{\langle\, V_{\alpha_1}(z_1)\, V_{\alpha_2}(z_2)\,\rangle}{\langle\, V_{\alpha_1}(z_1)\,\rangle\,\langle\, V_{\alpha_2}(z_2)\,\rangle} = e^{4\,\alpha_1\alpha_2\, g(z_1,z_2)}\, \frac{\left\langle e^{-\widetilde{S}_{\Delta,2,M}} \right\rangle_{z_1,\,z_2}}{\left\langle e^{-\widetilde{S}_{\Delta,1,M}} \right\rangle_{z_1} \left\langle e^{-\widetilde{S}_{\Delta,1,M}} \right\rangle_{z_2}}\ . \tag{79}$$

Notice that in this ratio the possible ambiguities λ_n in the subtraction terms of the geometric action, mentioned in Section 2, cancel out.

Again, we consider the cumulant expansion of $g^{\mathrm{g}}_{\alpha_1,\alpha_2}(\eta)$, which defines the functions $M^{\mathrm{g}}_{k_1,k_2}(\eta)$, as done in (72) for the standard approach.

From (79), we can get the functions $M^{\mathrm{g}}_{k_1,k_2}(\eta)$ within the geometric approach

$$M^{\mathrm{g}}_{1,1} = g(\eta) + \frac{1}{2^2}\left\{ -\left\langle \widetilde{S}^{(1,1)}_{\Delta,2,M} \right\rangle + \left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M}\, \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle - \left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M} \right\rangle \left\langle \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle \right\} \tag{80}$$

$$\begin{aligned}
M^{\mathrm{g}}_{2,1} = \frac{1}{2^3}\Bigg\{ &-\left\langle \widetilde{S}^{(2,1)}_{\Delta,2,M} \right\rangle + \left\langle \widetilde{S}^{(2,0)}_{\Delta,2,M}\, \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle + 2\left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M}\, \widetilde{S}^{(1,1)}_{\Delta,2,M} \right\rangle \\
&-\left\langle \left(\widetilde{S}^{(1,0)}_{\Delta,2,M}\right)^2 \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle - \left\langle \widetilde{S}^{(2,0)}_{\Delta,2,M} \right\rangle \left\langle \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle - 2\left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M} \right\rangle \left\langle \widetilde{S}^{(1,1)}_{\Delta,2,M} \right\rangle \\
&+\left\langle \left(\widetilde{S}^{(1,0)}_{\Delta,2,M}\right)^2 \right\rangle \left\langle \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle + 2\left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M}\, \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle \left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M} \right\rangle \\
&-2\left(\left\langle \widetilde{S}^{(1,0)}_{\Delta,2,M} \right\rangle\right)^2 \left\langle \widetilde{S}^{(0,1)}_{\Delta,2,M} \right\rangle \Bigg\}
\end{aligned} \tag{81}$$

$$M^{\mathrm{g}}_{3,1} = \ \dots$$

where

$$\widetilde{S}^{(k_1,k_2)}_{\Delta,2,M} = \frac{\partial^{k_1}}{\partial\alpha_1^{k_1}}\,\frac{\partial^{k_2}}{\partial\alpha_2^{k_2}}\,\widetilde{S}_{\Delta,2,M}\bigg|_{\alpha_1=\alpha_2=0}\ . \tag{82}$$

19

The graphs contributing to $M^{\mathrm{g}}_{k_1,k_2}(\eta)$ up to b^2_{g} included are shown below.

$$M^{\mathrm{g}}_{1,1} \quad = \quad \left[\; \bullet\text{-------}\bullet \; \right]$$

$$+ \; b^2_{\mathrm{g}} \left[\; \bullet\text{---}\bullet\!\bigcirc\!\bullet\text{---}\bullet \qquad \qquad \qquad \right]$$

$$+ \; O(b^4_{\mathrm{g}}) \; .$$

$$\tag{83}$$

$$M^{\mathrm{g}}_{2,1} \quad = \quad b_{\mathrm{g}} \left[\quad \right] \; + \; O(b^3_{\mathrm{g}}) \tag{84}$$

$$M^{\mathrm{g}}_{3,1} \quad = \quad b^2_{\mathrm{g}} \left[\quad \right] \; + \; O(b^4_{\mathrm{g}}) \; . \tag{85}$$

Comparing $M^{\mathrm{g}}_{k_1,k_2}(\eta)$ with the functions $M_{k_1,k_2}(\eta)$ obtained within the standard approach, we have found no differences for $M^{\mathrm{g}}_{2,1}(\eta)$ and $M^{\mathrm{g}}_{3,1}(\eta)$ up to $O(b^2)$ included, while a discrepancy with the standard approach arises comparing $M^{\mathrm{g}}_{1,1}(\eta)$ with $M_{1,1}(\eta)$. This difference is due to the fact that, using the Hadamard regularization, the second and the third graph of the $O(b^2_{\mathrm{g}})$ contribution to $M^{\mathrm{g}}_{1,1}(\eta)$ given in (83) sum up to zero, while, on the other hand, within the standard approach with ZZ regulator, the second and the third graph in the $O(b^2)$ contribution to $M_{1,1}(\eta)$ shown in (73) provide a non trivial function.
We have, up to order b^2

$$M^{\mathrm{g}}_{1,1}(\eta) \;\; = \;\; M_{1,1}(\eta) + b^2 \frac{2}{3} \left(-1 + \log(1-\eta) + \frac{\eta \, \log \eta}{1-\eta} + \frac{(1+\eta) \, \log(1-\eta) \, \log \eta}{2\,(1-\eta)} \right.$$
$$\left. + \frac{1+\eta}{1-\eta} \, \mathrm{Li}_2(1-\eta) \right) + O(b^4) \; . \tag{86}$$

It is interesting to compare such results with the exact formula conjectured in [3, 4, 7] for $G_{\alpha,-b/2}(z',z) = \langle\, V_\alpha(z') \, V_{-b/2}(z) \,\rangle$, by setting $\alpha_1 = \alpha$, $\alpha_2 = -b/2$, $z_1 = z'$ and $z_2 = z$.

The conjectured formula can be written equivalently in the upper half plane as [3]

$$G_{\alpha,-b/2}(\xi',\xi) = \frac{\left|\xi' - \bar{\xi}'\right|^{2\Delta_{-b/2}-2\Delta_{\alpha}}}{\left|\xi - \bar{\xi}'\right|^{4\Delta_{-b/2}}} \, U(\alpha)\, U(-b/2)\, (1-\eta)^{1+3b^2/2}\, g_{\alpha,-b/2}(\eta) \qquad (87)$$

and in the unit disk Δ representation as

$$G_{\alpha,-b/2}(z',z) = \frac{(1-z'\bar{z}')^{2\Delta_{-b/2}-2\Delta_{\alpha}}}{(1-z\bar{z}')^{4\Delta_{-b/2}}} \, U(\alpha)\, U(-b/2)\, (1-\eta)^{1+3b^2/2}\, g_{\alpha,-b/2}(\eta) \qquad (88)$$

where $U(\alpha)$ is the structure constant of the one point function conjectured in [3]. Notice that, to get (87) or equivalently (88), the ZZ boundary conditions (i.e. the cluster decay at large distance $B^{(-)}(\alpha) = U(\alpha)\, U(-b/2)$) have been used [3]. From

$$G_{\alpha,-b/2}(z',z) = \left\langle V_\alpha(z') \right\rangle \left\langle V_{-b/2}(z) \right\rangle \, g_{\alpha,-b/2}(\eta) \qquad (89)$$

where

$$g_{\alpha,-b/2}(\eta) = \eta^{\alpha b} \, {}_2F_1(1+b^2, 2\alpha b, 2+2b^2; 1-\eta) \qquad (90)$$

the cumulant expansion of $g_{\alpha,-b/2}(\eta)$ in α up to $O(b^3)$ is

$$\log\left[g_{\alpha,-b/2}(\eta)\right] =$$
$$= \alpha\left[b\left(-2g(\eta)\right) + b^3\left(-4 + \frac{\eta\log^2\eta}{1-\eta} + 2\frac{1+\eta}{1-\eta}\operatorname{Li}_2(1-\eta)\right) + O(b^5)\right]$$
$$+ \alpha^2\left[b^2\left(2 - 2\frac{\eta\log^2\eta}{(1-\eta)^2}\right) + O(b^4)\right]$$
$$+ \alpha^3\left[b^3\left(\frac{8}{3} + \frac{4}{3}\frac{\eta(1+\eta)\log^3\eta}{(1-\eta)^3}\right) + O(b^5)\right] + O(\alpha^4). \qquad (91)$$

It agrees perfectly with our perturbative results obtained within the standard approach and, as discussed above, it disagrees with the ones found within the geometric approach.

4   Invariance under background transformations

In order to improve our understanding of the two approaches, we shall examine in this section the dependence of the results on the choice of the background. We shall find that both the approaches are background independent.

The perturbative quantization of the lagrangian (14) with the boundary conditions of the pseudosphere has been carried through starting from the classical background solution ϕ_B^{cl}

and perturbatively expanding around it. This has been done both within the standard approach [3, 10] and within the geometric approach.

In this section, we shall examine the independence of the results on the choice of the background. We shall provide this calculation both at the formal level, by using the equation of motion for ϕ_M, and at the perturbative level, i.e. expanding the results perturbatively in the background field around ϕ_B^{cl} and verifying that these results are independent on such variations. This happens both within the standard approach and within the geometric approach, independently on the choice of the regulators. We have obtained the same results by employing the related Pauli-Villars regulator fields, described in Section 2.

First we give a formal proof of the background invariance within the standard approach. Using the background field method, i.e. decomposing the Liouville field ϕ as the sum of a background field ϕ_B and a quantum correction χ, the action on the pseudosphere with a generic background becomes [3]

$$S[\phi] = S_B[\phi_B] + S_\chi[\chi, \phi_B] \tag{92}$$

where

$$S_B[\phi_B] = \int_\Delta \left[\frac{1}{\pi} \partial_z \phi_B \, \partial_{\bar{z}} \phi_B + \mu e^{2b\phi_B} \right] d^2 z \tag{93}$$

and

$$S_\chi[\chi, \phi_B] = \int_\Delta \left[\frac{1}{\pi} \partial_z \chi \, \partial_{\bar{z}} \chi + \mu e^{2b\phi_B} \left(e^{2b\chi} - 1 \right) - \frac{2}{\pi} \chi \, \partial_z \partial_{\bar{z}} \phi_B \right] d^2 z \ . \tag{94}$$

Within the standard approach the correlation functions of the vertex operators are given by

$$\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_N}(z_N) \rangle = \frac{\int_{\mathcal{C}(\Delta)} \mathcal{D}[\chi] \, e^{-S_\chi[\chi, \phi_B] + \sum_n^N 2\alpha_n(\chi(z_n) + \phi_B(z_n))}}{\int_{\mathcal{C}(\Delta)} \mathcal{D}[\chi] \, e^{-S_\chi[\chi, \phi_B]}} = \frac{Z[J]}{Z[0]} \tag{95}$$

because the background contribution simplifies.

Under a variation $\delta\phi_B$ with compact support of the generic background field ϕ_B, we have

$$\delta \, \langle V_{\alpha_1}(z_1) \ldots V_{\alpha_N}(z_N) \rangle = \frac{\delta \, Z[J]}{Z[0]} - \frac{\delta \, Z[0]}{Z[0]} \frac{Z[J]}{Z[0]} \ . \tag{96}$$

From (95), we get

$$
\delta Z[J] = \int_{\mathcal{C}(\Delta)} \mathcal{D}[\chi] \int_{\Delta} \left[\frac{2}{\pi} \partial_z \partial_{\bar{z}} \chi - 2b\, \mu e^{2b\phi_B} \left(e^{2b\chi} - 1 \right) \right.
$$

$$
\left. + 2 \sum_{n=1}^{N} \alpha_n \delta^2(z - z_n) \right] \delta\phi_B \, d^2z \; e^{-S_\chi[\chi,\phi_B] + \sum_n^N 2\alpha_n(\chi(z_n) + \phi_B(z_n))}
\tag{97}
$$

that, due to the equation of motion in presence of sources for the field χ, becomes

$$
\delta Z[J] = Z[J] \int_{\Delta} \left[-\frac{2}{\pi} \partial_z \partial_{\bar{z}} \phi_B + 2b\, \mu e^{2b\phi_B} \right] \delta\phi_B \, d^2z
\tag{98}
$$

i.e. the variation of $Z[J]$ is given by a numerical factor that multiplies $Z[J]$. The same structure and the same numerical factor appear for the variation of $Z[0]$.

Thus, the first variation (96) of the correlation functions (95) under changes of the background vanishes. Since such a variation is null starting from a generic background field, we have the independence of the correlation functions from ϕ_B.

Obviously, the above general reasoning is only formal since the theory contains divergencies.

On the other hand, ZZ [3] proved the validity of the equations of motion on the background ϕ_B^{cl} up to order b^3 using their particular regulator and suggested their validity to all orders in the perturbation theory. Moreover starting from ϕ_B^{cl}, one can explicitly verify that G_1 and G_2 do not change under a variation $\delta\phi_B$ with compact support, to the first order in $\delta\phi_B$. We did it up to $O(b^3)$ included for G_1 and up to $O(b^2)$ included for G_2.

A completely similar reasoning works for the geometric approach.

The correlation functions of the Liouville vertex operators are given by

$$
\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_N}(z_N) \rangle = \frac{\displaystyle\int_{\mathcal{C}(\Delta)} \mathcal{D}[\phi_M] \, e^{-S_{\Delta,N}[\phi]}}{\displaystyle\int_{\mathcal{C}(\Delta)} \mathcal{D}[\phi_M] \, e^{-S_{\Delta,0}[\phi]}} = \frac{Z_g[J]}{Z_g[0]}
\tag{99}
$$

where $S_{\Delta,N}[\phi]$ is the geometric action (14) with a generic background satisfying the boundary conditions (7). Varying this action under a variation $\delta\phi_B$ of compact support of the background field, we get the variation of $Z_g[J]$

$$
\delta Z_g[J] = \int_{\mathcal{C}(\Delta)} \mathcal{D}[\phi_M] \int_{\Delta} \left[\frac{2}{\pi} \partial_z \partial_{\bar{z}} (\phi_M + g_0) \right.
$$

$$
\left. - 2b_g\, \mu_g e^{2b_g \phi_B} \left(e^{2b_g \phi_M} - 1 \right) \right] \delta\phi_B \, d^2z \, e^{-S_{\Delta,N}[\phi]}
$$

$$
+ Z_g[J] \left(2 \sum_{n=1}^{N} \alpha_n \delta\phi_B(z_n) \right)
\tag{100}
$$

23

that, through the equation of motion for ϕ_M, becomes

$$\delta\, Z_{\mathrm{g}}[\,J\,] \;=\; Z_{\mathrm{g}}[\,J\,] \int_\Delta \left[-\frac{2}{\pi}\, \partial_z \partial_{\bar{z}} \phi_B + 2 b_{\mathrm{g}} \mu_{\mathrm{g}} e^{2 b_{\mathrm{g}}\, \phi_B} \right] \delta\phi_B\, d^2 z \;. \tag{101}$$

Again, taking into account the variation of $Z_{\mathrm{g}}[\,0\,]$, we have to the first order in $\delta\phi_B$

$$\delta\, \langle\, V_{\alpha_1}(z_1) \dots V_{\alpha_N}(z_N)\,\rangle \;=\; \frac{\delta\, Z_{\mathrm{g}}[\,J\,]}{Z_{\mathrm{g}}[\,0\,]} - \frac{\delta\, Z_{\mathrm{g}}[\,0\,]}{Z_{\mathrm{g}}[\,0\,]}\, \frac{Z_{\mathrm{g}}[\,J\,]}{Z_{\mathrm{g}}[\,0\,]} \;=\; 0 \tag{102}$$

also in the geometric approach.

Within the regulated theory with the invariant regulator C in presence of the classical background ϕ_B^{cl}, we have verified the validity of the equations of motion up to $O(b_{\mathrm{g}}^4)$ included. Moreover one can verify that G_1^{g} and G_2^{g} do not vary respectively up to $O(b_{\mathrm{g}}^4)$ and $O(b_{\mathrm{g}}^3)$ included.

5   Geometric approach with ZZ regulator

From the results obtained in the previous sections, one could get the impression that the origin of the differences between the standard and geometric approach consists in the way used to introduce the sources.

An important point, already noticed in [11] and remarked in Section 2, is that, the Hadamard regularization within the geometric approach provides for the quantum conformal dimensions of the cosmological term $(\, 1 - b_{\mathrm{g}}^2, 1 - b_{\mathrm{g}}^2\,)$. The only way to find quantum dimensions $(\, 1, 1\,)$ for the cosmological term is to adopt also within the geometric approach the ZZ regulator [3]. With such a regulator, one obtains the same results of the standard approach to all orders. Indeed, there is a one to one correspondence between the graphs of the two approaches, except for the α^2 contribution to the quantum conformal dimensions, which is provided by a counterterm of the action within the geometric approach and by the one loop graph regularized through the ZZ regulator within the standard approach. Adopting the ZZ regulator within both approaches, the two treatments are identified for $b = b_{\mathrm{g}}$ and $\mu = \mu_{\mathrm{g}}$, and consequently

$$\left[\, \pi b_{\mathrm{g}}^2 \mu_{\mathrm{g}}\, \right]^{-\alpha_1/b_{\mathrm{g}}} e^{-2\alpha_1^2} \left\langle\, e^{-\widetilde{S}_{\Delta,1,M}}\, \right\rangle_{z_1} \;=\; \frac{U(\alpha_1)}{(\, 1 - z_1 \bar{z}_1\,)^{2 b \alpha_1}} \;=\; \left[\, \pi b^2 \mu\, \right]^{-\alpha_1/b} \frac{\left\langle\, e^{2\alpha_1 \chi(z_1)}\, \right\rangle}{(\, 1 - z_1 \bar{z}_1\,)^{2\alpha_1^2}} \tag{103}$$

where the mean value $\langle \dots \rangle_{z_1}$ is taken with respect to (28). The one point structure constant $U(\alpha_1)$ is given by the exact formula conjectured in [3] through the application of the bootstrap method.

Another difference between the two approaches is the asymptotic behaviour of $\langle\,\phi(z_1)\,\rangle$, defined as

$$\langle\,\phi(z_1)\,\rangle = \frac{1}{2}\,\frac{\partial}{\partial\alpha_1}\,\left\langle\,e^{2\alpha_1\phi(z_1)}\,\right\rangle\bigg|_{\alpha_1=0}\,. \tag{104}$$

Within the geometric approach with the C regulator one obtains

$$\langle\,\phi(z_1)\,\rangle \simeq -\frac{1}{2b_{\mathrm{g}}}\,\log\left(\,1 - z_1\bar{z}_1\,\right)^2 + \mathrm{const} \tag{105}$$

that is exactly the boundary condition (1) and (7) with $Q = 1/b_{\mathrm{g}}$, imposed respectively on the Liouville field ϕ and on the background field ϕ_B.

This does not happen within the standard approach with the ZZ regulator [3], where

$$\langle\,\phi(z_1)\,\rangle \simeq -\frac{1}{2}\left(\frac{1}{b} + b\right)\,\log\left(\,1 - z_1\bar{z}_1\,\right)^2 + \mathrm{const}\,. \tag{106}$$

The asymptotics (106) reproduces the boundary behaviour of the background field

$$\phi_{cl}(z) = -\frac{1}{2b}\,\log\left[\,\pi b^2\mu\,(\,1 - z\bar{z}\,)^2\,\right] \tag{107}$$

only qualitatively, i.e. with a different constant in front of the logarithm.

Thus, the main difference between the standard and the geometric approach is not so much in the way one uses to introduce the sources, but in the regulator chosen to make the divergent graphs finite, which reflects in itself the quantum nature of the cosmological term, deeply related the boundary behaviour of $\langle\,\phi(z_1)\,\rangle$.

Conclusions

In this paper we have compared the standard and the geometric approaches to quantum Liouville theory on the sphere and on the pseudosphere. Detailed perturbative calculations up to three loops have been performed on the pseudosphere in both approaches.

The geometric approach, even if invariant under $SU(1,1)$ group like the one of [2], gives different results with respect to standard approach [3]. A feature of the geometric approach, already noticed in [11], is that it provides the cosmological term with quantum conformal dimensions $(\,1 - b_{\mathrm{g}}^2, 1 - b_{\mathrm{g}}^2\,)$, while the standard approach gives $(\,1,1\,)$ for them. Moreover, while the perturbative calculations in the standard approach agree with the conformal bootstrap formulas of [3, 4, 7], this does not happen for the results obtained within the geometric approach. Both theories exhibit background invariance, yet they are definitely different.

A deeper analysis shows that the real difference between the two approaches does not concern the general setting, i.e. geometric action vs. standard sources approach, but it lies in the process of regularization. Indeed, as remarked in Section 5, adopting the ZZ regulator in the perturbative expansion of the geometric action, we have obtained the same results of the standard approach, even if the α^2 contributions to the quantum conformal dimensions have different formal origins within the two approaches.

Acknowledgments

Appendix

In this appendix, for sake of completeness, we shall give the transformation of the geometric action for the sphere [9, 11] into such a form that no limit process appears [12] and therefore suitable to perform perturbative computations. This form can be compared with the one derived for the pseudosphere in Section 1.

Again we write the Liouville field ϕ as a sum of a quantum field ϕ_M, a background field ϕ_B and a source field ϕ_0. Here we shall leave ϕ_B generic, except for the boundary conditions, and ϕ_0 shall be chosen as a solution of a linear equation with point sources.

The geometric action in presence of sources at the points $z_1, ..., z_N$ can be written, using the notation of [5], as follows

$$
\begin{aligned}
S_{\mathrm{P}^1, N}[\phi] \;=\; & \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left\{ \int_{\Gamma_{R,\varepsilon}} \left[\frac{1}{\pi} \partial_z \phi \, \partial_{\bar{z}} \phi + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}}\phi} \right] d^2 z \right. \\
& + \frac{Q}{2\pi i} \oint_{\partial \Gamma_R} \phi \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + Q^2 \log R^2 \\
& \left. - \frac{1}{2\pi i} \sum_{n=1}^{N} \alpha_n \oint_{\partial \gamma_n} \phi \left(\frac{dz}{z - z_n} - \frac{d\bar{z}}{\bar{z} - \bar{z}_n} \right) - \sum_{n=1}^{N} \alpha_n^2 \log \varepsilon_n^2 \right\}
\end{aligned}
\tag{108}
$$

where $d^2 z = (i dz \wedge d\bar{z})/2$ and the domains of integration are $\Gamma_R = \{|z| \leqslant R\}$, $\gamma_n = \{|z - z_n| \leqslant \epsilon_n\}$ and $\Gamma_{R,\varepsilon} = \Gamma_R \backslash \bigcup_n \gamma_n$.

The field ϕ is assumed regular at ∞ and to transform as [5]

$$
\phi(z, \bar{z}) \;\; \to \;\; \phi'(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{dw}{dz} \right|^2
\tag{109}
$$

26

under holomorphic coordinate transformations $z \to w(z)$, which implies

$$\phi(z, \bar{z}) \simeq -Q \log(z\bar{z}) + O(1) \qquad \text{for} \qquad |z| \to \infty . \tag{110}$$

The asymptotic behaviour of ϕ near the sources is

$$\phi(z, \bar{z}) \simeq -\alpha_n \log|z - z_n|^2 + O(1) \qquad \text{for} \qquad z \to z_n \tag{111}$$

as for the pseudosphere.

The geometric action in [11] is obtained for $Q = 1/b_\mathrm{g}$, but, to perform a comparison with the standard approach [5], we shall keep Q generic for a while.

From (108), one can extract the μ_g dependence of the correlation functions on the sphere by a proper constant shift of the Liouville field ϕ, as done in [5, 8, 7]. If we define the correlation functions on the sphere not dividing by $Z_\mathrm{g}[0]$, as suggested in [5], we obtain for such a dependence the following factor

$$\mu_\mathrm{g}^{(Q - \sum \alpha_n)/b_\mathrm{g}} . \tag{112}$$

The equation of motion derived from the geometric action (108) is the Liouville equation in presence of sources

$$\partial_z \partial_{\bar{z}} \phi = \pi b_\mathrm{g}\, \mu_\mathrm{g} e^{2b_\mathrm{g}\phi} - \pi \sum_{n=1}^{N} \alpha_n \delta^2(z - z_n) . \tag{113}$$

We write [12]

$$\phi = \phi_M + \phi_0 + \phi_B \tag{114}$$

where ϕ_B is the background field, regular on the whole plane

$$\phi_B = -Q \log(z\bar{z}) + c_B + O\left(\frac{1}{|z|}\right) \qquad |z| \to \infty \tag{115}$$

and the source field ϕ_0 is

$$\phi_0 = -\sum_{n=1}^{N} \alpha_n \log|z - z_n|^2 - \hat{\alpha}\, \phi_B + c_0 \tag{116}$$

i.e. the solution of

$$\partial_z \partial_{\bar{z}} \phi_0 = -\pi \sum_{n=1}^{N} \alpha_n \delta^2(z - z_n) - \hat{\alpha}\, \partial_z \partial_{\bar{z}} \phi_B \tag{117}$$

with $\hat{\alpha}$ given by

$$\hat{\alpha} = \frac{1}{Q} \sum_{n=1}^{N} \alpha_n \tag{118}$$

27

to have ϕ_0 going to a constant at ∞. As a consequence, also ϕ_M goes to a constant at ∞. The equation of motion for ϕ_M is

$$\partial_z \partial_{\bar{z}} \phi_M = \pi b_{\mathrm{g}} \mu_{\mathrm{g}} e^{2b_{\mathrm{g}}\phi} + (\hat{\alpha} - 1) \partial_z \partial_{\bar{z}} \phi_B . \tag{119}$$

Performing a number of integrations by part, the geometric action (108) can be rewritten as

$$S_{\mathrm{P}^1, N}[\phi] = S_{\mathrm{P}^1, B}[\phi_B] + S_{\mathrm{P}^1, N, M}[\phi_M, \phi_B] - \frac{2 - \hat{\alpha}}{\pi} \int_{\mathrm{P}^1} \phi_0 \, \partial_z \partial_{\bar{z}} \phi_B \, d^2 z$$

$$+ \sum_n^N \alpha_n \sum_{m \neq n}^N \alpha_m \, \log |z_n - z_m|^2 - \sum_n^N \alpha_n \Big(c_0 - \hat{\alpha} \, \phi_B(z_n) \Big) - 2 \sum_n^N \alpha_n \, \phi_B(z_n) \tag{120}$$

where $S_{\mathrm{P}^1, B}[\phi_B]$ is the background action [5]

$$S_{\mathrm{P}^1, B}[\phi_B] = \lim_{R \to \infty} \left\{ \int_{\Gamma_R} \left[\frac{1}{\pi} \partial_z \phi \, \partial_{\bar{z}} \phi_B + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}}\phi_B} \right] d^2 z \right.$$

$$\left. + \frac{Q}{2\pi i} \oint_{\partial \Gamma_R} \phi_B \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) + Q^2 \log R^2 \right\} \tag{121}$$

and $S_{\mathrm{P}^1, N, M}[\phi_M, \phi_B]$ is the action for the quantum field ϕ_M

$$S_{\mathrm{P}^1, N, M}[\phi_M, \phi_B] =$$
$$\int_{\mathrm{P}^1} \left[\frac{1}{\pi} \partial_z \phi_M \, \partial_{\bar{z}} \phi_M + \mu_{\mathrm{g}} e^{2b_{\mathrm{g}}\phi_B} \Big(e^{2b_{\mathrm{g}}(\phi_M + \phi_0)} - 1 \Big) - \frac{2(1 - \hat{\alpha})}{\pi} \phi_M \, \partial_z \partial_{\bar{z}} \phi_B \right] d^2 z . \tag{122}$$

Notice that, for $Q = 1/b_{\mathrm{g}}$, $e^{2b_{\mathrm{g}}\phi}$ is a $(1, 1)$ density at the classical level, which can be achieved by assigning to $e^{2b_{\mathrm{g}}\phi_B}$ a classical $(1, 1)$ nature and treating ϕ_0 and ϕ_M as scalars. The background action $S_{\mathrm{P}^1, B}[\phi_B]$, the action $S_{\mathrm{P}^1, N, M}[\phi_M, \phi_B]$ for the quantum field ϕ_M and the term

$$- \frac{2 - \hat{\alpha}}{\pi} \int_{\mathrm{P}^1} \phi_0 \, \partial_z \partial_{\bar{z}} \phi_B \, d^2 z \tag{123}$$

are invariant under a $SL(2, \mathbb{C})$ transformation

$$z \longrightarrow w = \frac{az + b}{cz + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \tag{124}$$

when $Q = 1/b_{\mathrm{g}}$.

Thus, the transformation properties of $S_{\mathrm{P}^1, N}[\phi]$ can be read from the second line of (120). Taking into account that, being ϕ_0 a scalar field, c_0 transforms like

$$c_0' = c_0 - \sum_{n=1}^N \alpha_n \, \log |cz_n + d|^2 \tag{125}$$

28

one obtains

$$S'_{\mathrm{P}^1, N}[\phi'] = S_{\mathrm{P}^1, N}[\phi] + \sum_{n=1}^{N} \alpha_n \left(Q - \alpha_n \right) \log \left| \frac{dw}{dz} \right|^2_{z=z_n}. \tag{126}$$

where $\log |dw/dz|^2_{z=z_n} = -2 \log |cz_n + d|^2$.

We remark that the geometric action (120) has the transformation property (126) under $SL(2, \mathbb{C})$ only for $Q = 1/b_{\mathrm{g}}$.

The crucial property of the geometric action is to provide, through (126) and the $SL(2, \mathbb{C})$ invariant measure, the quantum conformal dimensions

$$\Delta_\alpha = \alpha \left(Q - \alpha \right) = \alpha \left(\frac{1}{b_{\mathrm{g}}} - \alpha \right) \tag{127}$$

for the Liouville vertex operators $e^{2\alpha\phi(z)}$.

References

[1] T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. **48** (1982) 1309; E. Brateen, T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. **51** (1983) 19; Ann. Phys. **147** (1983) 365.

[2] E. D'Hoker and R. Jackiw, Phys. Rev. **D26** (1982) 3517; E. D'Hoker, D.Z. Freedman and R. Jackiw, Phys. Rev. **D28** (1983) 2583.

[3] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Liouville Field Theory on a Pseudo-sphere*, hep-th/0101152.

[4] V. Fateev, A.B. Zamolodchikov and Al.B. Zamolodchikov, *Boundary Liouville Field Theory I. Boundary State and Boundary Two-point Function*, hep-th/0001012.

[5] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. **B477** (1996) 577.

[6] H. Dorn and H.J. Otto, Nucl. Phys. **B429** (1994) 375.

[7] J. Teschner, Phys. Lett. **B363** (1995) 65; Class. Quant. Grav. **18** (2001) R153.

[8] M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991) 2051.

[9] L.A. Takhtajan, *Topics in Quantum Geometry of Riemann Surfaces: Two Dimensional Quantum Gravity*, Proc. Internat. School Phys. Enrico Fermi, 127, IOS, Amsterdam, 1996, hep-th/9409088.

[10] P. Menotti and E. Tonni, Phys. Lett. **B586** (2004) 425.

[11] L.A. Takhtajan, Mod. Phys. Lett. **A11** (1996) 93.

[12] L. Cantini, P. Menotti and D. Seminara, Nucl. Phys. **B638** (2002) 351.

[13] G. 't Hooft, in *Recent developments in gravitation*: Cargese, 1978, M. Levy and S. Deser (Eds.).

[14] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale Univ. Press, New Haven, Conn., 1923.

[15] P.R. Garabedian, *Partial Differential Equations*, John Wiley & Sons, New York, 1964.

[16] V. Moretti, Comm. Math. Phys. **201** (1999) 327.

[17] R. Courant and D. Hilbert, *Methods of Mathematical Physics* vol. II, Interscience Publishers Inc., New York, 1961.

[18] A.V. Kotikov, Phys. Lett. **B375** (1996) 240.